P216 Q2 2016 f Key

1- $f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

$= |x|$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$f(-x) = |-x| = |x| = f(x)$$

$\therefore$ $f(x)$ is even $\Rightarrow$ only $\cos nx$ in expansion

$\Rightarrow$ $b_n = 0$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$n \neq 0$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dy$$

$$= \frac{2}{\pi}\left[\frac{x}{n}\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left(\frac{1}{n^2}\cos nx\right)\Big|_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{\pi}{2}$$

$$\therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left((-1)^n - 1\right)\cos nx$$

$\therefore n = \text{odd} = 2k-1$

$$= \frac{\pi}{4} - \frac{4}{\pi}\sum_{k=1}^{\infty}\cos(2k-1)x$$

2- $f(x) = -\frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\sin n\pi x}{n}$

using parseval $(L=2)$

$$\frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f^2(x)\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

$a_n = 0$, $b_n = \frac{2}{n\pi} (-1)^{n+1}$

$f(x) = x \qquad -1 < x < 1$

$$\int_{-1}^{1} f^2(x)\, dx = \int_{-1}^{1} x^2\, dx$$

$$= \left. \frac{x^3}{3} \right|_{-1}^{1} = \frac{2}{3}$$

$$\therefore \frac{2}{3} = \sum_{n=1}^{\infty} \frac{4}{\pi^2} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

3 — $$\tilde{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|t| + i\omega t} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{at + i\omega t} \, dt + \int_{0}^{\infty} e^{-at + i\omega t} \, dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left. \frac{e^{(a+i\omega)t}}{a+i\omega} \right|_{-\infty}^{0} + \left. \frac{e^{(-a+i\omega)t}}{-a+i\omega} \right|_{0}^{\infty} \right] \qquad a > 0$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+i\omega} + \frac{1}{a-i\omega} \right]$$

$$= \frac{2a}{\sqrt{2\pi}} \frac{1}{a^2 + \omega^2}$$

4. $f(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases}$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega} \left(e^{i\omega a} - e^{-i\omega a}\right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}$$

$$\int_{-\infty}^{\infty} f^*(t) f(t)\, dt = \int_{-\infty}^{\infty} F^*(\omega) F(\omega)\, d\omega$$

$$\Rightarrow \int_{-a}^{a} dt = 2a = \int_{-\infty}^{\infty} \frac{2}{\pi} \frac{\sin^2 \omega a}{\omega^2}\, d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 \omega a}{\omega^2}\, d\omega = \pi a$$

Let $\quad w \to \dfrac{w}{a}$

$$\int_{-\infty}^{\infty} \frac{\sin^2 w}{\frac{w^2}{a^2}} \, \frac{dw}{a} = \pi a$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 w}{w^2} \, dw = \pi$$

5- $Ly = (1-x^2)y'' - xy' + n^2 y = 0$

$$\alpha = 1 - x^2$$

$$\beta = -x$$

L is Hermitian if

$$(\alpha w)' = \beta w$$

where $w(x)$ is weight function

$$(\alpha w)' = \frac{\beta}{\alpha}(\alpha w)$$

$$\therefore \frac{(\alpha w)'}{\alpha w} = \frac{\beta}{\alpha} = \frac{-x}{1-x^2}$$

$$\frac{d}{dx} \ln \alpha w = -\int \frac{x\,dx}{1-x^2}$$

$$= \frac{1}{2}\int \frac{d(1-x^2)}{1-x^2}$$

$\Rightarrow \ln \alpha w = \frac{1}{2} \ln (1-x^2)$

or $\quad \alpha w = \sqrt{1-x^2}$

$$w = \frac{\sqrt{1-x^2}}{1-x^2} = \frac{1}{\sqrt{1-x^2}}$$

Note: If $\mathcal{L}y = \alpha y'' + \beta y' + \gamma y$

then $\mathcal{L}$ is Hermitian if

$$(\alpha w)' = \beta w$$

6. $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

$P_0 = 1, \quad P_1 = x$

$P_1(-x) = -x = -P_1(x)$

$\therefore \quad P_n(-x) = (-1)^n P_n(x)$

is true for $n=1$

Assume it is true for

n: $P_n(-x) = (-1)^n P_n(x)$

then $P_{n+1}(-x) \overset{?}{=} (-1)^{n+1} P_{n+1}(x)$

Recursion relation for $-x$:

$$(2n+1)(-x)P_n(-x) = (n+1)P_{n+1}(-x) + n\,P_{n-1}(-y)$$

so

$$(2n+1)(-1)^{n+1} x P_n(x) = (n+1)P_{n+1}(-x) + n(-1)^{n-1} P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(-x) =$$

$$(-1)^{n+1}\left((2n+1)x P_n(x) - n P_{n-1}(x)\right)$$

$$= (-1)^{n+1}(n+1)P_{n+1}(x)$$

$$\Rightarrow \quad P_{n+1}(-x) = (-1)^{n+1} P_{n+1}(x)$$

$\Leftarrow$

7. Consider

$$\mathcal{L}Q_n = \frac{1}{w}\frac{d}{dx}\left(\alpha w \frac{dQ_n}{dx}\right) + \gamma Q_n = 0$$

$$\int_{-\infty}^{\infty} w\, Q_n^*(x)\, Q_m(x)\, dx = 0$$

$$\int_{-\infty}^{\infty} w\, Q_n^* \mathcal{L} Q_m(x)\, dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} dx\, w\, Q_n^* \left(\frac{1}{w}\frac{d}{dx}\left(\alpha \frac{dQ_m}{dx}\right) + \gamma Q_m\right)$$

$$= \int_{-\infty}^{\infty} Q_n^* \frac{d}{dx}\left(\alpha w \frac{dQ_m}{dx}\right) dx$$

$$= Q_n^* \left(\alpha \frac{dQ_m}{dx}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dQ_n^*}{dx} \cdot \alpha w \frac{dQ_m}{dx}$$

& $\phi_n' \phi_m'$ vanish at $\pm\infty$

$$\Rightarrow 0 = \int_{-\infty}^{\infty} (\alpha w)\, \phi_n' \, \phi_m' \, dx \qquad n \neq m$$

8. $f(\theta,\varphi) = \sin\theta\left(\sin^2\frac{\theta}{2}\cos\varphi + i\cos^2\frac{\theta}{2}\sin\varphi\right) + \sin^2\frac{\theta}{2}$

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 \Rightarrow \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta)$

$\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$

$\therefore f(\theta,\varphi) = \frac{1}{2}\Big[\sin\theta(1-\cos\theta)\cos\varphi$

$+ i\sin\theta(1+\cos\theta)\sin\varphi$

$+ (1-\cos\theta)\Big]$

$1 = \sqrt{4\pi}\, Y_0^0$

$\sin\theta\cos\varphi = +\sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^{1}\right)$

$\sin\theta\sin\varphi = \sqrt{\frac{2\pi}{3}}\, i\left(Y_1^{-1} + Y_1^{1}\right)$

$$\cos\theta = \sqrt{\frac{4\pi}{3}}\, Y_1^0$$

$$\sin\theta \cos\theta \cos\varphi = -\sqrt{\frac{15}{32\pi}}\left(Y_2^1 - Y_2^{-1}\right)$$

$$\sin\theta \cos\theta \sin\varphi = \sqrt{\frac{15}{32\pi}}\,\frac{1}{i}\left(Y_2^1 + Y_2^{-1}\right)$$

$$\therefore f(\theta,\varphi) = \frac{1}{2}\Bigg[\sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^1\right)$$

$$+\sqrt{\frac{15}{32\pi}}\left(Y_2^1 - Y_2^{-1}\right)$$

$$+\sqrt{\frac{2\pi}{3}}\,\frac{1}{i}\left(Y_1^{-1} + Y_1^1\right)$$

$$-\sqrt{\frac{15}{32\pi}}\,\frac{1}{i}\left(Y_2^1 + Y_2^{-1}\right)$$

$$+\sqrt{4\pi}\, Y_0^0$$

$$-\sqrt{\frac{3}{4\pi}}\, Y_1^0\Bigg]$$

9. $g(x,t) = e^{-t^2} e^{2tx}$

$$= \sum_{n=0}^{\infty} t^n H_n(x)$$

$$e^{-t^2+2tx} = e^{-t^2+2(-t)(-x)}$$

$$= \sum_{n=0}^{\infty} (-t)^n H_n(-x)$$

$$= \sum_{n=0}^{\infty} (-1)^n t^n H_n(-x)$$

$$\therefore \quad H_n(-x) = (-1)^n H_n(x)$$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2} - t^2 + 2tx} = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} H_n(x)\, dx\, t^n$$

$$= \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}(x-2t)^2 + t^2}$$

$$= e^{t^2} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}(x-2t)^2}$$

$x \to x - 2t$

$$= e^{t^2} \sqrt{2\pi}$$

$$\therefore \sum_{n=0}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} H_n(x) \right) t^n$$

$$= \sqrt{2\pi}\; e^{t^2}$$

since $e^{t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (t^2)^n$

is a power series in t^{2n}

only even powers appear

$$\therefore \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} H_{2n+1}(x)\, dx = 0$$

10) $xL_n^k = (2n+k+1)L_n^k$
$- (n+k)L_{n-1}^k$
$- (n+1)L_{n+1}^k$

$$\int_0^\infty e^{-x} x^{k+1} L_n^k L_n^k \, dx$$

$$= \int_0^\infty e^{-x} x^k (x L_n^k) L_n^k$$

$$= \int_0^\infty e^{-x} x^k \{ (2n+k+1)L_n^k - (n+k)L_{n-1}^k - (n+1)L_{n+1}^k \} L_n^k$$

by orthogonality $n \neq m$

$$\int e^{-x} x^k L_n^k L_m^k = 0$$

$$= \int_0^\infty e^{-x} x^k L_n^k L_n^k (2n+k+1)$$

use $\displaystyle\int_0^\infty e^{-x} x^k L_n^k L_n^k = \frac{(n+k)!}{n!}$

$$\Rightarrow \quad = (2n+k+1)\frac{(n+k)!}{n!}$$